

Deformed Geometric Algebra and Supersymmetric Quantum Mechanics

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Abstract

Deforming the algebraic structure of geometric algebra on the phase space with a Moyal product leads naturally to supersymmetric quantum mechanics in the star product formalism.

1 Introduction

Quantum mechanics has a natural description on the phase space in the star product formalism [1, 2]. Moreover spin and relativistic quantum mechanics can also be described in the star product formalism if one deforms pseudoclassical mechanics [3] with a fermionic star product. This was done in [4] and it was also shown that such a fermionic sector relates in supersymmetric quantum mechanics the supersymmetric partner systems.

One might then wonder which physical status pseudoclassical mechanics and its deformed version actually have. This question was solved in [5] where the relation of pseudoclassical mechanics and geometric algebra was established (for a comprehensive discussion of geometric algebra see for example [6]). It becomes then clear that the fermionic sector describes the geometric structure of the phase space in a multivector formalism. Furthermore the fermionic star product corresponds to the geometric product of geometric algebra that deforms Grassmann calculus into Clifford calculus [7].

Geometric algebra in its superanalytic formulation with the Clifford star product as the geometric product can then be used to describe the Hamilton formalism on the phase space. Moreover it appears natural to combine the fermionic star product that describes the geometric structure with the bosonic Moyal star product that makes this structure noncommutative. The result is a deformed, noncommutative version of geometric algebra. For geometric algebra on three-space the transition to noncommutativity induces an extra term that splits the system in a version with spin up and one with spin down, i.e. noncommutativity in geometric algebra on three-space transforms the Schrödinger Hamiltonian into the Pauli Hamiltonian [7].

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As will be shown below the transition to noncommutativity for geometric algebra on the phase space leads similarly to a split into two supersymmetric partner systems.

One can see the appearance of geometric algebra structures on the phase space also in a different way: Just as the factorization of the Klein-Gordon equation exhibits in Dirac theory the Clifford structure of space-time [8], the factorization of a Hamilton function into supercharges exhibits the Clifford structure of the phase space.

2 Geometric Algebra on the phase space

Geometric algebra was first used on the phase space to describe the Hamilton formalism in [9]. We will here restrict to the simplest case of a flat two-dimensional phase space and use the superanalytic formulation of geometric algebra. A point in the phase space is a vector or supernumber of Grassmann grade one:

$$z = z^i \zeta_i = q\boldsymbol{\eta} + p\boldsymbol{\rho}, \quad (2.1)$$

where the Grassmann variables $\zeta_1 = \boldsymbol{\eta}$ and $\zeta_2 = \boldsymbol{\rho}$ are the basis vectors of the two dimensional vector space. On this vector space the Clifford star product of two multivectors A and B is given by

$$A *_C B = A \exp \left[\eta_{ij} \overleftarrow{\frac{\partial}{\partial \zeta_i}} \overrightarrow{\frac{\partial}{\partial \zeta_j}} \right] B, \quad (2.2)$$

where $\eta_{ij} = \text{diag}(1, 1)$ is the euclidian metric on the vector space. Furthermore one has a closed two-form

$$\Omega = \frac{1}{2} \Omega_{ij} \zeta^i \zeta^j = \boldsymbol{\eta} \boldsymbol{\rho} = d q d p, \quad (2.3)$$

where Ω_{ij} is a non-degenerate, antisymmetric matrix [10] and $\boldsymbol{d} = \zeta^i \frac{\partial}{\partial z_i} = \boldsymbol{\eta} \frac{\partial}{\partial q} + \boldsymbol{\rho} \frac{\partial}{\partial p}$ is the nabla operator.

The euclidian scalar product of two vectors $\boldsymbol{a} = a^i \zeta_i$ and $\boldsymbol{b} = b^i \zeta_i$ is given by the scalar part of their star product (2.2), i.e. $\boldsymbol{a} \cdot \boldsymbol{b} = \langle \boldsymbol{a} *_C \boldsymbol{b} \rangle_0 = a^i b_i \eta_{ij}$ and with the two-form Ω the symplectic scalar product is given by

$$\boldsymbol{a} \cdot_{sy} \boldsymbol{b} = (\boldsymbol{b} \boldsymbol{a}) \cdot \Omega = \boldsymbol{a} \cdot (\Omega \cdot \boldsymbol{b}) = a^i \Omega_{ij} b^j. \quad (2.4)$$

Furthermore one can map with Ω a vector in a one-form according to $\boldsymbol{a}^\flat = \boldsymbol{a} \cdot \Omega$. The inverse map of a one-form into a vector can be described with the bivector

$$\boldsymbol{J} = \frac{1}{2} J^{ij} \zeta_i \zeta_j \quad (2.5)$$

so that the vector corresponding to a one-form $\boldsymbol{\omega} = \omega_i \zeta^i$ is given by $\boldsymbol{\omega}^\sharp = \boldsymbol{J} \cdot \boldsymbol{\omega}$. The map \sharp should be inverse to \flat , from which $J^{ij} = (\Omega_{ij}^{-1})^T = \Omega^{ji}$ follows. The Hamilton equations can then be written as:

$$\dot{z} = \boldsymbol{d}^\sharp H \quad (2.6)$$

and for the Poisson bracket one has

$$\{F, G\}_{PB} = F \overleftarrow{\boldsymbol{d}} \cdot_{sy} \overrightarrow{\boldsymbol{d}} G = J^{ab} \frac{\partial F}{\partial x^a} \frac{\partial G}{\partial x^b}. \quad (2.7)$$

3 Star-Factorization of the Hamilton Function

A Hamilton function can be written as the square of the vector

$$\mathbf{w} = W(q)\boldsymbol{\eta} + p\boldsymbol{\rho}, \quad (3.1)$$

where $W(q)$ is the Superpotential, one has then

$$H = \frac{1}{2}\mathbf{w} *_C \mathbf{w} = \frac{1}{2}\mathbf{w} \cdot \mathbf{w} = \frac{1}{2} [p^2 + W^2(q)] \quad (3.2)$$

and in holomorphic coordinates $B = \frac{1}{\sqrt{2}}(W(q) + ip)$, $\bar{B} = \frac{1}{\sqrt{2}}(W(q) - ip)$ and $\mathbf{f} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta} + i\boldsymbol{\rho})$, $\bar{\mathbf{f}} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta} - i\boldsymbol{\rho})$ one obtains

$$\mathbf{w} = B\bar{\mathbf{f}} + \bar{B}\mathbf{f} = \mathbf{Q}_+ + \mathbf{Q}_- \quad (3.3)$$

and $H = B\bar{B}$.

Up to now the coefficients were commuting quantities, but one can go over to the noncommutative or quantum case by demanding that the coefficients have to be multiplied by the Moyal product

$$f *_M g = f \exp \left[\frac{i\hbar}{2} \left(\overleftarrow{\frac{\partial}{\partial q}} \overrightarrow{\frac{\partial}{\partial p}} - \overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial q}} \right) \right] g. \quad (3.4)$$

In this case the square of \mathbf{w} is no longer a scalar, but one has an bivector valued extra term

$$H_S = \frac{1}{2}\mathbf{w} *_M \mathbf{w} = \frac{1}{2} [(W(q) *_M W(q))(\boldsymbol{\eta} *_C \boldsymbol{\eta}) + (W(q) *_M p)(\boldsymbol{\eta} *_C \boldsymbol{\rho}) \\ + (p *_M W(q))(\boldsymbol{\rho} *_C \boldsymbol{\eta}) + (p *_M p)(\boldsymbol{\rho} *_C \boldsymbol{\rho})] \quad (3.5)$$

$$= \frac{1}{2} [p^2 + W^2(q)] + \frac{\hbar}{2} \frac{\partial W(q)}{\partial q} \frac{1}{i} \boldsymbol{\eta} \boldsymbol{\rho}. \quad (3.6)$$

The next thing one has to notice is that $\boldsymbol{\eta}$, $\boldsymbol{\rho}$ and $-i\boldsymbol{\eta}\boldsymbol{\rho}$ fulfill under the Clifford star product the Pauli algebra, i.e. one has for the star commutators $[A, B]_{*C} = A *_C B - B *_C A$ and anticommutators $\{A, B\}_{*C} = A *_C B + B *_C A$ of these real basis elements of the two dimensional Clifford algebra:

$$[\boldsymbol{\eta}, \boldsymbol{\rho}]_{*C} = 2i\boldsymbol{\eta}\boldsymbol{\rho}, \quad [\boldsymbol{\eta}, -i\boldsymbol{\eta}\boldsymbol{\rho}]_{*C} = -2i\boldsymbol{\rho}, \quad [\boldsymbol{\rho}, -i\boldsymbol{\eta}\boldsymbol{\rho}]_{*C} = 2i\boldsymbol{\eta} \quad (3.7)$$

$$\text{and } \{\boldsymbol{\eta}, \boldsymbol{\eta}\}_{*C} = \{\boldsymbol{\rho}, \boldsymbol{\rho}\}_{*C} = \{-i\boldsymbol{\eta}\boldsymbol{\rho}, -i\boldsymbol{\eta}\boldsymbol{\rho}\}_{*C} = 2, \quad (3.8)$$

while the other star commutators and star anticommutators vanish. This means that $\boldsymbol{\eta}$, $\boldsymbol{\rho}$ and $-i\boldsymbol{\eta}\boldsymbol{\rho}$ would be represented in a tuple representation by the Pauli matrices, so that H_S is the supersymmetric Hamiltonian. Furthermore for the holomorphic basis vectors $\mathbf{f} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta} + i\boldsymbol{\rho})$ and $\bar{\mathbf{f}} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta} - i\boldsymbol{\rho})$ one has the tuple representation

$$\frac{1}{\sqrt{2}}\mathbf{f} \cong \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\bar{\mathbf{f}} \cong \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3.9)$$

The two eigen-multivectors of $-i\boldsymbol{\eta}\boldsymbol{\rho}$ are $\pi_{\pm}^{(C)} = \frac{1}{2}(1 \mp i\boldsymbol{\eta}\boldsymbol{\rho})$, i.e. for these multivectors one has

$$-i\boldsymbol{\eta}\boldsymbol{\rho} *_C \pi_{\pm}^{(C)} = \pm \pi_{\pm}^{(C)}. \quad (3.10)$$

In the star product formalism these multivectors are fermionic Wigner functions and as such they are projectors:

$$\pi_{\pm}^{(C)} *_C \pi_{\pm}^{(C)} = \pi_{\pm}^{(C)} \quad \text{and} \quad \pi_+^{(C)} *_C \pi_-^{(C)} = \pi_-^{(C)} *_C \pi_+^{(C)} = 0, \quad (3.11)$$

while in geometric algebra these multivectors are related to spinors [11]. The holomorphic basis vectors $\frac{1}{\sqrt{2}}\mathbf{f}$ and $\frac{1}{\sqrt{2}}\bar{\mathbf{f}}$ serve here as lowering and raising operators, i.e. one has

$$\bar{\mathbf{f}} *_C \pi_+^{(C)} *_C \mathbf{f} = 2\pi_-^{(C)} \quad \text{and} \quad \mathbf{f} *_C \pi_-^{(C)} *_C \bar{\mathbf{f}} = 2\pi_+^{(C)}, \quad (3.12)$$

while the other combinations give zero.

With the multivectors $\pi_{\pm}^{(C)}$ the supersymmetric Hamilton function (3.6) can then be written as

$$\begin{aligned} H_S &= \frac{1}{2} \left[p^2 + W^2(q) - \hbar \frac{\partial W(q)}{\partial q} \right] \left(\frac{1}{2} - \frac{i}{2} \boldsymbol{\eta} \boldsymbol{\rho} \right) \\ &\quad + \frac{1}{2} \left[p^2 + W^2(q) + \hbar \frac{\partial W(q)}{\partial q} \right] \left(\frac{1}{2} + \frac{i}{2} \boldsymbol{\eta} \boldsymbol{\rho} \right) \end{aligned} \quad (3.13)$$

$$= H_1 \pi_+^{(C)} + H_2 \pi_-^{(C)}. \quad (3.14)$$

From (3.11) it is then clear that the Moyal-Clifford star eigenfunctions of H_S are a product of $\pi_+^{(C)}$ and Moyal star eigenfunctions of H_1 or products of $\pi_-^{(C)}$ and Moyal star eigenfunctions of H_2 . The Moyal star eigenfunctions for supersymmetric partner potentials were for examples discussed in [12].

The vectors \mathbf{Q}_{\pm} defined in (3.3) fulfill

$$\mathbf{Q}_{\pm} *_M \mathbf{Q}_{\pm} = 0, \quad \mathbf{Q}_- *_M \mathbf{Q}_+ = H_1 \pi_+^{(C)}, \quad \mathbf{Q}_+ *_M \mathbf{Q}_- = H_2 \pi_-^{(C)} \quad (3.15)$$

so that H_S can be written as

$$H_S = \frac{1}{2} \{ \mathbf{Q}_+, \mathbf{Q}_- \} *_M, \quad (3.16)$$

and with (3.15) one has $[\mathbf{Q}_+, H_S] *_M = [\mathbf{Q}_-, H_S] *_M = 0$. Defining eventually

$$\mathbf{Q}_1 = \mathbf{Q}_+ + \mathbf{Q}_- \quad \text{and} \quad \mathbf{Q}_2 = -i(\mathbf{Q}_+ - \mathbf{Q}_-) \quad (3.17)$$

the supersymmetric Hamilton function factorizes as

$$H_S = \frac{1}{2} \mathbf{Q}_1 *_M \mathbf{Q}_1 = \frac{1}{2} \mathbf{Q}_2 *_M \mathbf{Q}_2. \quad (3.18)$$

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